# Notes By Ibrahim Al Balushi

# Lecture 16

## Dirichlet Problem for Poisson eq.

$$\begin{cases} \Delta u = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \tag{1}$$

If $\Delta v = f$, then $w = u - v$ satisfies

$$\begin{cases} \Delta w = 0 & in\ \Omega \\ w = g - v\ on\ \partial\Omega \end{cases}$$

We need $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Our candidate

$$u(x) = \int_\Omega E(x-y) f(y) dy$$

But $f \in C(\Omega)$ is not sufficient; $\exists u \notin C^1(\Omega)$ $s.t$ $\Delta u \in C(\Omega)$. This leads to the notion of Holder continuous spaces. In particular, we will show that the integrability and boundedness of $f$ can only guarantee a function $u \in C^1$ in the integral expression above. It will be established that a necessary condition for $u \in C^2$ which solves $\Delta u = f$ is that $f$ must not only be continuous, but also Holder continuous. The uniqueness will of $u$, should it exist, will also depend on the boundary of $\Omega$, as described in the discussion above.

**Definition 1.** *Let $f : \Omega \to \mathbb{R},\ y \in \Omega,\ 0 < \alpha < 1$. The function $f$ is said to be Holder continuous at $y$ with respect to exponent $\alpha$ if*

$$\sup_{x\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty. \tag{2}$$

*We write, $f \in C^{k,\alpha}(\Omega)$, where $k$ denoted order of continuous differentiability.*
*We define a seminorm by*

$$|f|_{C^\alpha(\Omega)} = \sup_{x,y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \tag{3}$$

*and we define a norm on $C^{k,\alpha}$ by*

$$\|f\|_{C^{k,\alpha}(\Omega)} = \|f\|_{C^k(\Omega)} + |f|_{C^\alpha(\Omega)} \tag{4}$$

For convenience we will adopt the following notation:

$$\|f\|_{k,\alpha;\Omega} := \|f\|_{C^{k,\alpha}(\Omega)}$$

Note that if $f \in C^{0,\alpha}$ then we write $f \in C^\alpha$. Meanwhile if $f \in C^{k,0}$ then we write $f \in C^k$.

## Newtonian Potential

**Definition 2.** *For any integrable $f$ on domain $\Omega$ we define the Newtonian potential of $f$ as a function $w$ defined by*

$$w(x) = \int_{\Omega} E(x-y)f(y)dy, \tag{5}$$

*where $E(x-y)$ defines the fundamental solution for the Laplacian.*

Suppose $f$ is defined to be the right hand side of the Laplace equation, then $w(x)$ would define the solution for $\Delta u = f$, ie $w := u$. By this definition, we may deduce information and estimates on derivatives of $u$ by studying the behaviour of integral (5). These estimates will form basis for Schauder estimates and the extension of potential theory from the Laplace equation to general linear elliptic PDE's.

**Theorem 1.** *Let $f$ be bounded and integrable in $\Omega$, and let $w$ be the Newtonian potential of $f$. Then $w \in C^1(\overline{\Omega})$ and for any $x \in \Omega$,*

$$\partial_i w(x) = \int_{\Omega} \partial_i E(x-y)f(y)dy, \qquad i = 1, ..., n. \tag{6}$$

*Proof.* Choose $\eta$ such that $\eta \in C^\infty(\mathbb{R})$, $\eta \geq 0$, $\eta = 0$ *on* $[0,1]$ $\eta = 1$ *on* $[2, \infty)$ and define $\eta_\epsilon(t) = \eta\left(\frac{t}{\epsilon}\right)$, for some $\epsilon > 0$,

$$w_\epsilon(x) = \int_{\Omega} E(x-y)\eta_\epsilon(|x-y|)f(y)dy \tag{7}$$

is clearly in $C^1(\overline{\Omega})$. now,

$$|w_\epsilon(x) - w(x)| = \left| \int_{\Omega} E(x-y)[\eta_\epsilon(|x-y|) - 1]f(y)dy \right| \tag{8}$$

$$\leq \|f\|_\infty \int_{\Omega} |E(x-y)||\eta_\epsilon(|x-y|) - 1|dy \tag{9}$$

$$\tag{10}$$

note that $\eta_\epsilon(|x-y|) - 1 = \eta(\epsilon^{-1}|x-y|) - 1 = 0$ whenever $|x-y| \geq 2\epsilon$ so we have

$$= \|f\|_\infty \int_{|x-y| \leq 2\epsilon} |E(x-y)|dy$$

by change of variables $|x-y| = r$ we have

$$= \|f\|_\infty \int_0^{2\epsilon} \frac{C}{r^{n-2}} r^{n-1} dr \leq C\epsilon^2 \|f\|_\infty$$

Hence $w_\epsilon \to w$ uniformly in $\Omega$. Constant $C$ is a result of higher dimensional polar integration; $\int_\Phi \varphi d\phi$, $\varphi$ being change of variables Jacobian determinant. Now let

$$v(x) = \int_{\Omega} \partial_i E(x-y)f(y)dy \tag{11}$$

We aim to assert the final statement. Naively, one may immediately attempt to employ Liebnez rule by differentiating under the integral, however $E(x-y)$ is not continuous nor bounded whenever $x-y=0$. This is essentially why we make use of $\eta_\epsilon(t)$ where we cut out that region for some $\epsilon>0$. This then allows us to invoke Liebnez rule and showing that the result holds at the limits.

$$|\partial_i w_\epsilon(x) - v_i(x)| = \left| \frac{\partial}{\partial x_i} \int\limits_\Omega E(x-y)\eta_\epsilon(|x-y|)f(y)dy - \int\limits_\Omega \frac{\partial E(x-y)}{\partial x_i} f(y)dy \right| \tag{12}$$

$$= \left| \int\limits_\Omega \frac{\partial E(x-y)}{\partial x_i}\eta_\epsilon(|x-y|)f(y)dy + \int\limits_\Omega E(x-y)\frac{\partial \eta_\epsilon(|x-y|)}{\partial x_i}\frac{x_i-y_i}{|x-y|}f(y)dy \right. \tag{13}$$

$$\left. - \int\limits_\Omega \frac{\partial E(x-y)}{\partial x_i} f(y)dy \right| \tag{14}$$

$$= \left| \int\limits_\Omega \frac{\partial E_y}{\partial x_i}[\eta_\epsilon - 1]f(y)dy + \int\limits_\Omega E_y \partial_i \eta_\epsilon(|x-y|)\frac{x_i-y_i}{|x-y|}f(y)dy \right| \tag{15}$$

$$\leq C\|f\|_\infty \int\limits_0^{2\epsilon} \frac{r^{n-1}}{r^{n-1}}dr + \int\limits_\epsilon^{2\epsilon} E_y \frac{x_i-y_i}{|x-y|}f(y)dy \tag{16}$$

$$\implies \partial_i w_\epsilon \to v_i \qquad uniformly$$

□

A stronger result occurs should $f$ be Holder continuous on $\Omega$ as well. In particular, the Newtonian potential becomes $C^2$ instead of $C^1$ by the conditions above.

**Theorem 2.** *Let $f$ be bounded and locally Holder continuous with $\alpha \in [0,1]$ in $\Omega$, and let $w$ be the Newtonian potential of $f$. Then $w \in C^2(\Omega)$, $\Delta w = f$ in $\Omega$, and for any $x \in \Omega$*

$$\partial_i\partial_j w(x) = \int\limits_{\Omega'} \partial_{ij}E(x-y)(f(y)-f(x))dy - f(x)\int\limits_{\partial\Omega'} \partial_i E(x-y)\nu_j(y)\ dS_y, \qquad i,j=1,...,n. \tag{17}$$

*where $\Omega' \supset \Omega$ for which the divergence theorem holds and $f$ is extended to vanish outside $\Omega$*

*Proof.* The proof of regularity is similar to the previous theorem, however the crucial step is when we use Holder continuity. Define the RHS of (17) by $\mu(x)$ and set $v = \partial_i w$ so for $\epsilon > 0$ define

$$v_\epsilon(x) = \int\limits_\Omega \partial_i E(x-y)\eta_\epsilon(|x-y|)f(y)dy \tag{18}$$

We can write the difference

$$|\mu(x) - \partial_j v_\epsilon(x)| = \left| \int\limits_{|x-y|\leq 2\epsilon} \frac{\partial}{\partial x_j}\left\{[1-\eta_\epsilon(|x-y|)]\frac{\partial}{\partial x_i}E(x-y)\right\}(f(y)-f(x))dy \right| \tag{19}$$

$$Holder\ continuity\ implies\ |f(x)-f(y)| \leq [f]_{\alpha;x}|x-y|^\alpha \tag{20}$$

$$\leq [f]_{\alpha;x} \int\limits_{|x-y|\leq 2\epsilon} \frac{\partial}{\partial x_j}\left\{[1-\eta_\epsilon(|x-y|)]\frac{\partial}{\partial x_i}E(x-y)\right\}|x-y|^\alpha dy \tag{21}$$

To assert the solvability of $\Delta w = f$ we will set $\Omega' = B_R(x)$ where $R$ is sufficiently large in (17) we have

$$\Delta w = \sum_i \int\limits_{B_R(x)} \partial_i^2 E(x-y)(f(y)-f(x))dy - f(x) \int\limits_{\partial B_R(x)} \partial_i E(x-y)\nu_i \, dS_y \tag{22}$$

$$= \int\limits_{B_R(x)/B_\delta(x)} \underbrace{\Delta E(x-y)}_{=0}[f(y)-f(x)] \tag{23}$$

$$+ \underbrace{\int\limits_{B_\delta(x)}}_{\substack{\longrightarrow 0 \\ as\ \delta\to 0}} \sum_i \partial_i^2 E(x-y)[f(y)-f(x)] - f(x) \int\limits_{\partial B_R(x)} \partial_\nu E_y \, dS_y \tag{24}$$

$$= -f(x) \int\limits_{\partial B_R(x)} \partial_\nu E_y \tag{25}$$

$$\textit{taking } \nu_i = -\frac{x_i - y_i}{|x-y|} \textit{ we have,} \tag{26}$$

$$= \frac{f(x)}{|S^{n-1}|R^{n-1}} \int\limits_{|x-y|=R} dS_y \tag{27}$$

$$= f(x) \tag{28}$$

□

**Lemma 1.** *$u \in C^1(\mathbb{R}^n)$ if $\in C(\Omega) \cap L^\infty(\Omega)$. Moreover, $\|u\|_{C^1} \leq C\|f\|_\infty$.*

*Proof.*

$$u_\epsilon \to u$$
$$\partial_i u_\epsilon \to v_i$$

$$u(x+he_i) - u(x) = u_\epsilon(x+he_i) - u_\epsilon(x) + o_\epsilon(1) \tag{29}$$
$$= h\partial_i u\epsilon(x+h'e_i) + o_\epsilon(1) \tag{30}$$
$$= hv_i(x+h'e_i) + ho_\epsilon(1) + o\epsilon(1) \tag{31}$$
$$= hv_i(x) + ho_\epsilon(1) + o_\epsilon(1) \tag{32}$$
$$\tag{33}$$

□

$$v_{i,\epsilon}(x) = \int\limits_\Omega \underbrace{\partial_i E(x-y)\eta_\epsilon(|x-y|)}_{N} f(y)dy$$

$v_{i,\epsilon} \to v_i$ uniformly.

$$\partial_i v_{i\epsilon}(x) = \int_B \partial_i N(f(y) - f(x))dy + f(x) \int_B \partial_i N dy \qquad F = Ne_i \to divF = \frac{\partial N}{\partial y_i} = -\frac{\partial N}{\partial x_j} \tag{34}$$

$$= \int_B [f(y) - f(x)]\partial_i N + f(x) \int_{\partial B} \left( -\frac{\partial N}{\partial \nu} \right) \tag{35}$$

$$for\ B\ large\ enough \tag{36}$$

$$= \int_B [f(y) - f(x)]\partial_i N - f(x) \int_{\partial B} v_j d_i E \tag{37}$$

$$v_{ij}(x) = \int_B d_j d_i E(x-y)[f(y) - f(x)]dy - \int_{\partial B} f(x)\nu_j \partial_i E(x-y)dy^{n-1}$$

$$\underbrace{\partial_i v_{i,\epsilon} - v_{ij}(x)}_{w} = \int_B [f(y) - f(x)] \underbrace{\partial_j [(\eta_\epsilon(|x-y|) - 1)\partial_i E(x-y)]}_{A} dy$$

$$A = \eta_\epsilon \frac{x_i - y_i}{|x-y|}$$

$$|A| \leq \frac{C}{\epsilon r^{n-1}} + \frac{C}{r^2}$$

suppose $|f(y) - f(x)| \leq w(|x-y|)$

$$|D| \leq \int_0^{2\epsilon} \frac{Cw(r)r^{n-1}dr}{\epsilon r^{n-1}} + \int_0^{2\epsilon} \frac{Cw(r)r^{n-1}}{r^n} dr$$

$$\leq \int \frac{Cw(r)dr}{\epsilon} + \int \frac{Cw(r)}{r} dr \to 0 as \epsilon \to 0.$$

**Definition 3.** *$f \in C(\Omega)$ is called Dini continuous if $\forall K \subset \Omega$ compact, $\exists w : \mathbb{R}_o \to \mathbb{R}_o$ integrable s.t*

$$|f(x) - f(y)| \leq w(|x-y|) \qquad \forall x, y \in K$$

*and that $\exists R > 0,\ \int_0^R w(r)r^{-1}dr < \infty$*

Example Holder continuity $w(r) = r^\alpha,\ \alpha \in (0,1)$.

**Theorem 3.** *$u \in C^2(\Omega)$ if $f$ is dini continuous in $\Omega$*

# Lecture 17

## Last time

$\Omega$ bounded domain, $f \in C(\overline{\Omega})$

$$u(x) = \int_\Omega E(x-y)f(y)dy \implies u \in C^1(\mathbb{R}^n)$$

If $f$ is Dini in $\Omega$, then

$$\partial_j \partial_i u(x) = \int\limits_B [f(y) - f(x)] K(x-y) dy - f(x) \int\limits_{\partial B} \nu_j \partial_i E(x-y) d^{n-1} y$$

$B \supseteq \Omega,\ K = \partial_j \partial_i E$, extend $f$ by 0 outside of $\Omega$. $u \in C^2(\Omega)$

**Theorem 4.** *If $f$ is Dini in $\Omega$, then $u \in C^2(\Omega) \cup C^1(\mathbb{R}^n)$ and $\Delta u = f$ in $\Omega$. In addition, if $f \in C^{0,\alpha}(B)$ and $\overline{\Omega} \subset B$, then*

$$\|u\|_{C^2} \leq C \|f\|_{C^\alpha(B)}.$$

*Proof.* We have

$$\Delta u(x) = \sum_i \partial_i \partial_i u(x) \tag{38}$$

$$= \int\limits_{\partial B_R(x)} (\quad) \underbrace{\sum_i \partial_i \partial_i E(x-y)}_{=0} dy - f(x) \int\limits_{\partial B_R(x)} \sum_i \nu_i \partial_i E(x-y) d^{n-1} y \tag{39}$$

$$\partial_i E(x-y) = \frac{1}{|S^{n-1}|} \frac{x_i - y_i}{|x-y|^n} = -\frac{\nu_i}{|S^{n-1}||x-y|^{n-1}}$$

where $-\nu_i = \frac{x_i - y_i}{|x-y|}$

$$I_2 = \int\limits_{|x-y|=R} \frac{d^{n-1}}{|S^{n-1}||x-y|^{n-1}} = -\frac{|S^{n-1}|R^{n-1}}{|S^{n-1}|R^{n-1}} = 1 \implies \Delta u(x) = f(x)$$

bound for :

$$|\partial_j \partial_i u(x)| \leq \int\limits_B [f]_{\alpha,x} |x-y|^\alpha |x-y|^{-n} dy + |f(x)| C$$

$[f]_{\alpha,x} := \sup_{y \in B} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$ and $|f|_{C^\alpha(B)} = \sup_{x \in B} [f]_{\alpha,x}$ we have

$$\|f\|_{C^\alpha(B)} = \|f\|_{C^\alpha(B)} + |f|_{C^\alpha(B)}$$

□

**Theorem 5.** *$\Omega$ bounded, $\partial\Omega$ regular everywhere, $f \in C(\overline{\Omega})$ Dini in $\Omega$, $g \in C(\partial\Omega)$. Then $\exists! u \in C^2(\Omega) \cap C(\overline{\Omega})$ s.t*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

*Proof.*

$$v(x) = \int_\Omega E(x-y) f(y) dy \implies v \in C^2(\Omega) \cap C^1(\mathbb{R}^n)$$

. Then solve

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = g - v & \text{on } \partial\Omega \end{cases}$$

$\implies u = v + w \in C^2(\Omega) \cap C(\overline{\Omega})$

$$\begin{cases} \Delta u = \Delta v + \Delta w = f & in\ \Omega \\ u = v + (g - v) = g & on\ \partial\Omega \end{cases}$$

□

## Second Order Elliptic Equations

$$Lu = \sum_{ij} a_{ij}\partial_i\partial_j u + \sum_j b_j\partial_j u + b_0 u$$

$a_{ij}, b_k \in C(\overline{\Omega})$. $A = (a_{ij})$ is Symmetric Positive definite : $\lambda_1|\xi|^2 \leq \xi^T A\xi \leq \lambda_n|\xi|^2$ where $\forall\xi \in \mathbb{R}^n$ and $\lambda_1$ and $\lambda_n$ positive constants. Uniformly elliptic.
Given the conditions above, at each $x_0 \in \Omega$, $\exists Q \in O(n)$ s.t

$$QA(x_0)Q^T = \Lambda \quad diagonal$$

$\Lambda = diag(\lambda_1, ..., \lambda_n)$, $\lambda_i > 0$.

## Method of Continuity

**Theorem 6.** *$X, Y$ Banach spaces, $A_0, A_1 : X \to Y$ bounded linear operators, $A_t = (1-t)A_0 + tA_1 : X - \to Y$. Assume $\exists C > 0$ s.t*

$$\|u\|_X \leq C\|A_t u\|_Y \quad \forall u \in X$$

*$A_0$ is surjective $\implies$ $A_1$ surjective.*

*Proof.* Injectivity implies $A_t u = A_t v \implies \|u - v\|_X \leq 0 \implies u = v$, $A_t$ injective. Assume $A_t$ surjective. Let $s \in [0, 1]$. $A_s u = f$ given some arbitrary $f \in Y$. We want to use Banach fixed point theorem. That is, construct mapping $\phi : X \to X$ s.t $\phi u = u$.

$$A_t u = f - A_s u + A_t u = f + (t - s)(A_1 - A_0)u$$

$$u = A_t^{-1} f + (t - s)A_t^{-1}(A_1 - A_0)u =: \phi u$$

$$\|\phi(u - v)\|_X = \|(t - s)A_t^{-1}(A_1 - A_0)(u - v)\|_X \tag{40}$$

$$\leq \underbrace{|t - s|\|A_t^{-1}\|\|A - 1 - A_0\|}_{want<1}\|u - v\|_X \tag{41}$$

$$\tag{42}$$

sufficient condition:

$$|t - s| < \frac{1}{2}\|A_1 - A_0\|^{-1}\|A_t^{-1}\|^{-1}$$

$$\|A_t^{-1}\| = \sup_{f\in Y}\|A_t^{-1}f\|_X/\|f\|_Y \leq \sup_f C\|f\|_Y/\|f\|_Y = C$$

□

## Schauder Estimates

We will use method of continuity with $X = C_o^{2,\alpha}(\overline{\Omega}),\ Y = C^{0,\alpha}(\Omega)$.

$$\begin{cases} Lu = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

<u>idea:</u> Localization, freezing coefficients, estimate the error $(L - A_0)u$, partition of unity :

$$\bigcup B_i \supseteq \Omega.\ \chi_i \in C^{\infty}, 1]),\ \sum_i \chi_i = 1\ in\ \overline{\Omega}$$

$L \to A_0 = A(x_0)$ from var to constant coeff. We now estimate

$$\|u\|_X = \|\sum_i \chi_i u\|_X \leq \sum_i \|\chi_i u\|_X \leq C \sum_i \|A(x_i)\chi_i u\|_Y$$

$$\leq C \sum_i \|L\chi_i u\|_Y + C \sum \|[A(x_i) - A]\chi_i u\|_Y + \|B\chi_i u\|_Y$$

$$\leq C \sum_i \underbrace{\|\chi_i L u\| Y}_{\leq C\|Lu\|_Y} + \|(L\chi_i - \chi_i L)u\|_Y + \|[A(x_i) - A]\chi_i u\|_Y + \|B\chi_i u\|_Y$$

# Lecture 18

## Motivation from Last time

$$L = \sum_{ij} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_j + b_0$$

Assume $a_{ij}, b_k \in C^{\alpha}(\overline{\Omega})$. $\exists \alpha > 0\ s.t\ \alpha\xi^T\xi \leq \xi^T A\xi,\ \forall \xi \in \mathbb{R}^n$. Aiming for

$$\|u\|_{C^{2,\alpha}} \leq C\|[(1-t)\Delta + tL]u\|_{C^{\alpha}}$$

(continuity of operators)

$$A_t = (1-t)I + tA \implies \xi^T A_t\xi \geq (1-t)\xi^T\xi + \alpha t\xi^T\xi \geq \min(1,\alpha)\xi^T\xi.$$

sufficient is to show

$$\|u\|_{C^{2,\alpha}} \leq C\|Lu\|_{C^{\alpha}}$$

with C mildly dependence on $L$. Recall

$$u(x) = \int_{\Omega} E(x-y)f(y)dy, \qquad f \in C^{\alpha}(\overline{B}),\ \overline{\Omega} \subset B$$

$$\partial_i\partial_j u(x) = \int_{\Omega} [f(y) - f(x)]\partial_i\partial_j E(x-y)dy - f(x)\int_{\partial B} \nu_j\partial_i E(x-y)d^{n-1}y$$

and $u \in C(\Omega) \cap C^1(\mathbb{R}^n),\ \Delta u = f\ in\ \Omega,\ \|u\|_{C^2(\Omega)} \leq C\|f\|_{C^{\beta}(B)}$

$$\partial_i\partial_j u(x') - \partial_{ij}u(x) = f(x)\int_{\partial B} \nu_j\partial_i[E(x'-y) - E(x-y)] + [f(x') - f(x)]\int_{\partial B} \nu_i\partial_i E(x'-y) +$$

consider the midpoint between $x$ and $\overline{x}$ and take ball $\delta = |x - x'|$ $\overline{x} = \frac{x+x'}{2}$

$$+\int_{B_\delta(\overline{x})} [f(y) - f(x)]K(x-y)dy + \qquad (x \to \overline{x})$$

$\partial_i \partial_j E(x-y) =: K(x-y)$

$$+[f(x') - f(x)] \int_{B/B_\delta(\overline{x})} K(x-y)dy + \int_{B/B_\delta(\overline{x})} [f(y) - f(x')][K(x-y) - K(x'-y)]dy$$

$$\Omega = B_R(z),\ B = B_{2R}(z)$$

we find bounds

$$|I_1| \leq |x' - x| \int_{\partial B} |\partial \partial_i E(x-y)| d^{n-1}y \leq \frac{|x - x'}{R^n} |\partial B|$$

where R is the distance between $\partial\Omega$ and $\partial B$ where $B \supset \Omega$ and $y \in \partial B$

$$\leq C \frac{|x - x'|}{R} \leq C' \delta^\alpha$$

$$|I_2| \leq C,$$

$$|I_3| \leq \int_{B_\delta(\overline{x})} [f]_{x,\alpha} |x - y|^{\alpha - n}$$

if $y$ was outside ball then $|y - x| \geq C|y - \overline{x}|$

$$\leq C[f]_{x,\alpha} \int_{B_{3\delta/2}(x)} |x - y|^{\alpha - n}$$

$$\leq C[f]_{x,\alpha} \delta^\alpha$$

$$|I_4| \leq C[f]_{x',\alpha} \delta^\alpha$$

$$|I_5| \leq C + |\int_{B_\delta(\overline{x})} \nu_j \underbrace{\partial_i E(x-y)}_{C\delta^{1-n}} d^{n-1}y| \leq C$$

$$|I_6| \leq [f]_{x',\alpha} \int |y - x'|^\alpha |x - x'||\partial K(\xi - y)|dy$$

$$\leq [f]_{x',\alpha} |x - x'| \int_{B/B_\delta(\overline{x})} \frac{|y - x'|^\alpha}{|\xi - y|^{n+1}} dy$$

$$C[f]_{x',\alpha} |x - x'| \int |y - \overline{x}|^{\alpha - n - 1} dy$$

$$C[f]_{x',\alpha} \delta \int_\delta^\infty r^{\alpha - n - 1} r^{n-1} dr$$

$$\leq C[f]_{x',\alpha} \delta \delta^{\alpha - 1}$$

so

$$|\partial_{ij} u(x) - \partial_{ij} u(x')| \leq C(|f(x)|\delta^\alpha + [f]_{x,\alpha}\delta^\alpha + [f]_{x',\alpha}\delta^\alpha)$$

$$\implies |u|_{C^{2,\alpha}(\Omega)} \leq C\|f\|_{C^\alpha(B)}$$

## up to boundary

Half annuluis on upper half space.

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_n > 0\}$$
$$B_r^+ = B_r \cap \mathbb{R}^n_+$$
$$B_1 = B_R,\ B_2 = B_{2R}$$
$$\Sigma = B_2 \cap \{x_n = 0\}$$

$x \in \overline{B_1^+}$

$$\partial_i \partial_j u(x) = \int_{B_2^+} [f(y) - f(x)] K(x-y) dy - f(x) \int_{\partial B_2^+} \nu_j \partial_i E(x-y) d^{n-1} y$$

$\nu_j$ has nonzero comp upwards on sigma so

$$j \neq n: \int_\Sigma ... = 0$$

same as the interior case.

$$i \neq n: \ by\ symetry \int \nu_j \partial_i = \int \nu_i \partial_j$$

$$i = j = n: \ \Delta u = f \implies \partial_n^2 u = f - \sum_{i=1}^{n-1} \partial_i^2 u$$

$$|\partial_n^2 u|_{C^\alpha(B_1^+)} \leq |f|_{C^\alpha(B_1^+)} + \sum_{i=1}^{n-1} |\partial_i^2 u|_{C^\alpha(B_1^+)}$$
$$\leq C\|f\|_{C^\alpha(B_2^+)} \implies |u|_{C^{2,\alpha}(B_1^+)} \leq C\|f\|_{C^\alpha(B_2^+)}$$

**Theorem 7.** $u \in C^2(B_2) \cap C(\overline{B_2}),\ f \in C^\alpha(\overline{B_2}),\ \Delta u = f\ in\ B_2$

$$\implies u \in C^{2,\alpha}(B_1)\ and \|u\|_{C^{2,\alpha}}(B_1) \leq C(\|u\|_{C(B_2)} + \|f\|_{C^\alpha(B_2)})$$

*Proof.*

$$w = E * f \implies \Delta w = f\ in B_2,\ v = u - w \implies \Delta v = 0\ in\ B_2.$$
$$\|w\|_C^{2,\alpha}(B_1) \leq C\|f\|_{C^\alpha(B_2)}$$
$$\|v\|_{C^{2,\alpha}}(B_1) \leq C\|v\|_{C(B_2)} \leq C\|u\|_{C(B_2)} + C\|w\|_{C(B_2)}$$

which $C\|w\|_{C(B_2)} \leq C\|f\|_{C(B_2)}$ ∎

**Lemma 2.**

$$X \subset Y \subset Z,$$

*Banach spaces that are contiuously embedded :*

$$\|u\|_Y \leq C_1 \|u\|_X \qquad \|u\|_Z \leq C_2 \|u\|_Y$$

*with $X$ compactly embbded in $Y$:*

$$\{u \in X | \|u\|_X \leq 1\}\ is\ compact\ in\ Y$$

$$\implies \forall \epsilon > 0, \exists C(\epsilon) \geq 0 st$$
$$\|u\|_Y \leq \epsilon \|u\|_X + C(\epsilon) \|u\|_Z \qquad \forall u \in X$$

# Lecture 19

# Last time

**Theorem 8.** *$u \in C^2(B_2) \cap C(\overline{B}_2),\ f \in C^\alpha(B_2),\ \Delta u = f$ in $B_2$; $B_i := B_{iR}$*

$$\implies u \in C^{2,\alpha}(B_1),\ \|u\|_{C^{2,\alpha}(B_1)} \leq C(\|u\|_{C(B_2)} + \|f\|_{C^\alpha(B_2)})$$

*where $C$ depends on $R$.*

$$|u|_{C^{2,\alpha}(B_R)} \leq C(R^{-2-\alpha}\|u\|_{C(B_{2R})} + R^{-\alpha}\|f\|_{C(B_{2B})} + \|f\|_{C^\alpha(B_{2R})})$$

Analogy: dimensional transformation

$$x = \tilde{R}\tilde{x} \qquad R\tilde{\Omega} = \Omega \qquad \tilde{u}(\tilde{x}) = u(R\tilde{x})$$

$$|\tilde{u}|_{C^\alpha(\tilde{\Omega})} = \sup_{\tilde{\Omega}} = \frac{|u(R\tilde{x}) - u(R\tilde{y})|}{|\tilde{x} - \tilde{y}|^\alpha} = \sup_{\Omega} \frac{|u(x) - u(y)|}{R^\alpha |x - y|^\alpha} = R^{+\alpha}|u|_{C^{\alpha}(\Omega)}$$

$$|u|_{C^{2,\alpha}(B_R)} = R^{-2-\alpha}|\tilde{u}|_{C^{2,\alpha}(B_1)} \leq Const \cdot R^{-2-\alpha}(\|\tilde{u}\|_{C(B_2)} + R^2\|f\|_{C(B_2)} + R^2|f|_{C^\alpha(B_2)})$$

$$= \cdot R^{-2-\alpha}(\|u\|_{C(B_{2R})} + R^2\|f\|_{C(B_{2R})} + R^{2+\alpha}|f|_{C^\alpha(B_{2R})})$$

Observation:

$$x = \phi(\tilde{x}) \ coord\ transformation$$

$$|\tilde{x} - \tilde{y}| \lesssim .|\phi(\tilde{x}) - \phi(\tilde{y})| \lesssim .|\tilde{x} - \tilde{y}| \implies |u|_{C^\alpha(\Omega)} \simeq |u|_{C^\alpha(\tilde{\Omega})}$$

$$X \simeq Y \implies X \lesssim Y \lesssim X$$

We provied for upper half annulus that $u = E * f$

$$\|u\|_{C^{2,\alpha}(B_1^+)} \lesssim \|f\|_{C^\alpha(B_2^+)}$$

**Theorem 9.** *$u \in C^2(B_2^+) \cap C(\overline{B_2^+}),\ f \in C^\alpha(B_2^+),\ \Delta u = f$ in $B_2^+,\ u = 0$ on $\Sigma$*

$$\implies u \in C^{2,\alpha}(B_1^+) \ and\ \|u\|_{C^{2,\alpha}(B_1^+)} \lesssim \|u\|_{C(B_2^+)} + \|f\|_{C^\alpha(B_2^+)}$$

*Proof.*

$$w(x) = \int\limits_{B_2^+} E(x-y)f(y)dy - \int\limits_{B_2^+} E(x-y^*)f(y)dy \qquad y^*\ reflection\ about\ \Sigma-(base\ of\ upper\ half\ annulus)$$

$$\implies \Delta w = f \ in\ B_1^+,\ w = 0 \ on\ \Sigma$$

$$v = u - w \implies \Delta v = 0 \ in\ B_1^+,\ v = 0 \ on\ \Sigma.$$

$$\|v\|_{C^{2,\alpha}}(B_1^+) \lesssim \|v\|_{C(B-2)} \lesssim \underbrace{\|u\|_{C(B_2)}}_{\leq con\|u\|_{C(B_2^+)}} + \|w\|_{C(B_2)}$$

$$take\ f^*(y) = f(y^*)\ for\ y \in B_2^- \qquad f*(y) = f(y)\ y \in B_2^+ \cup \Sigma$$

$$w(x) = 2\int_{B_2^+} E(x-y)f(y)dy - \int_{B_2} E(x-y)f^*(y)dy$$

$$\|f^*\|_{C(B_2)} \leq \|f\|_{C(B_2^+)}, \qquad \|f^*\|_{C^\alpha(B_2)} \leq \|f\|_{C^\alpha(B_2^+)}$$

$$\frac{|f^*(x)-f^*(y)|}{|x-y|^\alpha} \leq \frac{|f(y)-f(x)|}{|x-y|^\alpha}\ for\ x,y \in B_2^+ \qquad \leq \frac{|f(x)-f(y^*)|}{|x-y^*|^\alpha}\ x \in B_2^+,\ y \in B_2^-$$

□

**Lemma 3.** *X compactly embedded in Y continuously embedded in Z Banach*

$$\forall \epsilon > 0,\ \exists C(\epsilon) > 0\ s.t\ \|u\|_Y \leq \epsilon\|u\|_X + C(\epsilon)\|u\|_Z \qquad \forall u \in C$$

*Proof.* Let $\epsilon > 0$ be given. $\exists\{u_n\} \supset X,\ \|u_n\|_X = 1,\ st\ \|u_n\|_Y > \epsilon + n\|u\|_Z \qquad n \in \mathbb{N}$

$$u_n \to u \quad in\ Y,\ \|u_n\|_Z < \frac{1}{n}\|u\|_Y \to 0 \implies \|u-u_n\|_Z \leq C\|u-u_n\|_Y \to 0 \implies u = 0\ but$$

$$\|u\|_Y \geq \|u_n\|_Y - \|u-u_n\|_Y > \epsilon - \|u-u_n\|_Y\ contradiction.$$

□

$$|uv(x)-uv(y)| \leq |u(x)v(x)-u(y)v(y)|+|u(y)v(x)-u(y)v(y)| \leq |u|_{C^\alpha}|x-y|^\alpha|v(x)|+|u(y)||v|_{C^\alpha}|x-y|^\alpha$$

$$\implies |uv|_{C^\alpha} \leq |u|_{C^\alpha}|v|_{C^0} + |u|_{C^0}|v|_{C^\alpha}$$

# Global Schauder Estimates

$$L = \underbrace{\sum a_{ij}\partial_i\partial_j + \sum b_i\partial_i + b_i}_{A+B}, \qquad a_{ij}, b_k \in C^\alpha(\Omega).$$

Let $y \in \Omega$ interior. $A_y = \sum a_{ij}(y)\partial_i\partial_j$. $r > 0$ is such that $B_{2r} \subset \Omega$

$$\|u\|_{C^{2,\alpha}(B_1)} \lesssim r^{-2}\|u\|_{C(B_2)} + r^{-\alpha}\|A_y u\|_{C(b_2)} + |A_y u|_{C^\alpha(B_2)}$$

$$\|A_y u\|_{C(B_2)} \leq \|Lu\|_{C(B_2)} + \|(A_y - A)u\|_{C(B_2)} + \|Bu\|_{C(B_2)}$$

$$\|Bu\|_C \lesssim \|b_i \partial u\|_C \lesssim \|b\|_C\|u\|_{C^1}$$

$$\|(A_y - A)u\|_{C(B_2)} \lesssim \|(a_{ij}(y) - a_{ij})\partial^2 u\|_{C(B_2)} \lesssim \|a_{ij}\|_C\|u\|_{C^2(B_2)}$$

$$a_{ij}(y) - a_{ij}) \lesssim r^\alpha\|a_{ij}\|_{C^\alpha}$$

$$|A_y u|_{C^\alpha(B_2)} \lesssim \|Lu\|_{C^\alpha(B_2)} + \|(A_y - A)u\|_{C^\alpha} + \|B_u\|_{C^\alpha}$$

$$\|B_u\|_{C^\alpha} \leq \|b_k\|_{C^\alpha}\|u\|_{C^{1,\alpha}}$$

$$\|(A_y - A)u\|_{C^\alpha} \lesssim \|a_{ij}(y) - a_{ij}\|_{C^\alpha}\|u\|_{C^2} + \|a_{ij}(y) - a_{ij}\|_C\|u\|_{C^{2,\alpha}} \lesssim \|u\|_{C^2} + r^\alpha\|u\|_{C^{2,\alpha}(B_2)}$$

choose r small enough, $u \in C_0^{2,\alpha}(B_1) \implies$

$$\|u\|_{C^{2,\alpha}} \lesssim \|Lu\|_{C^\alpha} + \|u\|_{C^2}$$

$$\|u\|_{C^2} \leq \epsilon\|u\|_{C^{2,\alpha}} + C(\epsilon)\|u\|_{C^0}$$

# Lecture 20

# Last time

$$L = \sum a_{ij}\partial_i\partial_j + \sum b_i\partial_i + b_0 \qquad a_{ij},\ b_k \in C^\alpha(\overline{\Omega})$$

**Theorem 10.** *$y \in \Omega$ $\exists r > 0$ small such that*

$$\|u\|_{C^{2,\alpha}_o(B_r(y))} \lesssim \|Lu\|_{C^\alpha B_r(y)} + \|u\|_{C^2(B_r(y))}$$

*for all $u \in C^{2,\alpha}_o(B_1^+)$.*

**Lemma 4.**

$$\|u\|_{C^{2,\alpha}(B_1^+)} \lesssim \|\Delta u\|_{C^\alpha(B_2^+)} + \|u\|_{C(B_2^+)}$$

*$u \in C^{2,\alpha}(B_2^+) \cap C(\overline{B_2^+})$, $u = 0$ on $\Sigma$. : upper hald annuli with base $\Sigma$*

**Definition 4.** *we say $\partial\Omega \in C^{k,\alpha}$ is $\forall u \in \partial\Omega$ $\exists U_y$ nbhd of $y$ and $\exists\phi : U_y \to B_1$ s.t $\phi \in C^{2,\alpha}(U_y)$, $\exists\phi^{-1} : B_1 \to U_y$, $\phi^{-1} \in C^{2,\alpha}(B_1)$ and $\phi(U_y \cap \Omega) = B_1^+$, $\phi(U_y \cap \partial\Omega) = \partial B - 1^+ \backslash \partial B_1$.*

**Theorem 11.** *$\partial\Omega \in C^{2,\alpha}$, $y \in \partial\Omega$ $\exists U$ nbhd of $y$ such that*

$$\|u\|_{C^{2,\alpha}(U^+)} \lesssim \|Lu\|_{C^\alpha(U^+)} + \|u\|_{C^2(U^+)}$$

*$U^+ = U \cap \Omega$ $\forall u \in C^{2,\alpha}_0(U)$, $u = 0$ on $U \cap \partial\Omega$*

*Proof.* By definition, $\exists\phi : U_y \to B_1$; $U_r = \phi^{-1}(B_r)$ $(r < 1)$

$$u \in C^{2,\alpha}_o(U_r),\ u = 0 \ on \partial\Omega$$

$$\|u\|_{C^{2,\alpha}(U_r^+)} \lesssim \|u\|_{C^{2\alpha}(B_r^+)} \lesssim \|u\|_{C(B_r^+)} + |A_y u|_{C^\alpha}(B_r^+) + r^{-\alpha}\|A_y u\|_{C(B_r^+)}$$

$\to$for $r > 0$ sufficnetly small

$$\|u\|_{C^{2,\alpha}(U_r^+)} \lesssim \|Lu\|_{C^\alpha(U_r^+)} + \|u\|_{C^2(U_r^+)}$$

□

**Theorem 12** (Shauder 1934)**.** *$\Omega \subset \mathbb{R}^n$ bdd domain, $\partial\Omega \in C^{2,\alpha}$, $\exists C > 0$ s.t*

$$\|u\|_{C^{2,\alpha}}(\Omega) \le C(\|Lu\|_{C^\alpha(\Omega)} + \|u\|_{C(\Omega)}) \qquad \forall u \in C^{2,\alpha}(\overline{\Omega}) \qquad u = 0 \ on\ \partial\Omega$$

*Proof.* $\forall y \in \overline{\Omega}$ let $U_y$ be nbhd of $y$ s.t local estimates holds $\{U_y\}$ open cover of $\overline{\Omega}$. choose a finite subcover $\{U_i\}$. we have $\{\chi_i\}$ (partion of unitiy ) subordinate to $\{U_i\}$

$$\chi_i : \mathbb{R}^n \to [0,1] \qquad \chi_i \in C^\infty_o(U_i) \qquad \sum \chi_i \quad on \overline{\Omega}$$

$U_i^+ = U_i$

$$\|u\|_{C^{2,\alpha}(\Omega)} = \|\sum \chi_i u\|_{C^{2,\alpha}(\Omega)} \le \sum \|\chi_i u\|_{C^{2,\alpha}(U_i^+)} + \|\chi_i u\|_{C^2(U_i^+)}$$

$$\lesssim \sum(\|Lu\|_{C^\alpha(U_i^+)} + \|u\|^2_C(U_i^+)) \lesssim \sum_i(\|Lu\|_{C^\alpha(\Omega)} + \|u\|_{C^2(\Omega)}) \lesssim \|Lu\|_{C^\alpha(\Omega)} + \|u\|_{C^2(\Omega)}$$

$$\|\chi_i u\|_{C^2(U_i^+)} \lesssim \|u\|_{C^2(U_i^+)}$$
$$L\chi_i u = \chi_i Lu + (L\chi_i - \chi_i L)u \ (firstorder)$$
$$\|L\chi u\|_{C^\alpha} \lesssim \|\chi_i Lu\|_{C^\alpha} + \|u\|_{C^{1,\alpha}} \lesssim \|Lu\|_{C^\alpha}$$

$$\implies \|u\|_{C^{2,\alpha}(\Omega)} \leq C(\|Lu\|_{C^\alpha(\Omega)} + \|u\|_{C^2(\Omega)})$$
$$\|u\|_{C^{2,\alpha}}(\Omega) \leq \epsilon\|u\|_{C^{2,\alpha}} + C(\epsilon)\|u\|_C$$

□

for method of continuity we need $\|u\|_{C^{2,\alpha}} \lesssim \|Lu\|_{C^\alpha}$. In general the extra term $+\|u\|_C$ is unavoidable because for example

$$\begin{cases} \Delta u + ku = 0 \\ u = 0 \end{cases}$$

has non trivial solution.

Maximum principal:

$$b_0 \qquad Lu \geq 0 \qquad in\ \Omega \qquad u \in C^2(\Omega) \cap C(\overline{\Omega})$$
$$\implies \sup_\Omega u \leq \sup_{\partial\Omega} u$$

*Proof.* $Lu > 0$ in $\Omega$. $\Omega \ni y$ s.t $u(y) = \sup_\Omega u$ hence $Lu \leq 0 (\Delta u \leq 0)$. Define $u_\epsilon = u + \epsilon e^{\lambda x}$ □

**Corollary 1.** $b_0 \leq 0\ Lu \geq 0 \implies u \leq max\{0, \sup_{\partial\Omega} u\}$

*Proof.* $\Omega^+ = \{x \in \Omega : u(x) > 0\}$

$$L_o u = \sum a_{ij}\partial_i\partial_j u + \sum b_i \partial_i u \geq -b_0 u \geq 0$$

in $\Omega^+$

$$\implies \sup_{\Omega^+} u \leq \sup_{\partial\Omega^+} u \leq \max\{0, \sup_{\partial\Omega} u\}$$

□

**Corollary 2.** $b_0 \leq 0,\ Lu = f \implies |u| \leq \sup_{\partial\Omega}|u| + C\|f\|_\infty$.

*Proof.* Choose $\lambda > 0$ such that

$$\underbrace{L_0 e^{\lambda x_1}}_{a_{11}\lambda e^{\lambda x_1} + b\lambda e^{\lambda x_1}} \geq \gamma e^{\lambda x_1} \geq \gamma$$

for some $\gamma > 0$. Let

$$v(x) = \sup_{\partial\Omega}|u| + \frac{e^{\lambda d} - e^{\lambda x_1}}{\gamma}\|f\|_\infty$$
$$Lv \leq L_0 u \leq -\|f\|_\infty$$
$$L(u+v) \leq f - \|f\|_\infty \leq 0 \qquad u + v \geq 0 \qquad on\ \Omega$$
$$L(-u+v) \leq -f - \|f\|_\infty \leq 0 \qquad -u + v \geq 0 \qquad on\ \partial\Omega$$

$\implies |u| \leq v = \sup_{\partial\Omega}|u| + C\|f\|_\infty$. □

If $b_0 \leq 0 :\ \|u\|_{C^{2,\alpha}(\Omega)} \lesssim \|Lu\|_{C^\alpha(\Omega)}$

**Theorem 13** (Shauder). *$b_0 \leq 0$ $\partial\Omega \in C^{2,\alpha}$ $f \in C^\alpha(\overline{\Omega})$. Then there exists unique $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*moreover, $u \in C^{2,\alpha}(\overline{\Omega})$ and $\|u\|_{C^\alpha(\Omega)} \lesssim \|f\|_{C^\alpha(\Omega)}$*

*Proof.*

$$X = \{u \in C^{2,\alpha}(\Omega) |\ u = 0 \text{ on } \partial\Omega\}$$

$$y \in C^\alpha(\Omega)$$

□

# Lecture 21

$$L = \sum a_{ij}\partial_i\partial_j + \sum b_i\partial_i + b_0$$

where $\{a_{ij}\}_{ij}$ symmetric positive matrix.

**Theorem 14** (Schauder Regularity Thm). *Let $a_{ij},\ b_k \in C^\alpha(\overline{\Omega}),\ \partial\Omega \in C^{2,\alpha},\ f \in C^\alpha(\overline{\Omega}),\ b_0 \leq 0.$*

$$\implies \exists! u \in C^2(\Omega) \cap C(\overline{\Omega}) \ s.t$$

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

*In addition, $\|u\|_{2,\alpha;\Omega} \lesssim \|f\|_{\alpha;\Omega}$*

Implicitly assume:

**Theorem 15** (kellogg). *$\partial\Omega \in C^{2,\alpha},\ f \in C^\alpha(\overline{\Omega}),\ u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\implies u \in C^{2,\alpha}(\overline{\Omega})$$

*sketch.* Strategy:- first prove for balls then apply Schauder regularity theorem for balls. □

## Inhomogeneous Dirichlet Boundary Condition

**Definition 5.** *We say $g \in C^{2,\alpha}(\partial\Omega)$ if $\exists v \in C^{2,\alpha}(\overline{\Omega})$ s.t*

$$v|_{\partial\Omega} = g \quad \text{and} \quad \|g\|_{2,\alpha,\partial\Omega} := \{\|v\|_{2,\alpha;\Omega} :\ v|_{\partial\Omega} = g\} \tag{43}$$

**Corollary 3.** *$f \in C^\alpha(\overline{\Omega}),\ g \in C^{2,\alpha}(\partial\Omega) \implies \exists! u \in C^2(\Omega) \cap C(\overline{\Omega})$ s.t*

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \|u\|_{2,\alpha;\Omega} \lesssim \|f\|_{\alpha;\Omega} + \|g\|_{2,\alpha;\Omega}$$

*Proof.* Let $v \in C^{2,\alpha}(\overline{\Omega})$ $s.t$ $v|_{\partial\Omega} = g,$ $\|v\|_{2,\alpha} \le 2\|g\|_{2,\alpha}$

$$\begin{cases} Lw = f - Lv & in\ \Omega \\ w = 0 & on\ \partial\Omega \end{cases} \qquad and \qquad \|w\|_{2,\alpha} \lesssim \|f\|_\alpha + \|v\|_{2,\alpha}$$

$u = v + w$

$$\|u\|_{2,\alpha} \le \|v\|_{2,\alpha} + \|w\|_{2,\alpha}$$

□

Meaning:

$$F(u) = (Lu, u|_{\partial\Omega}), \qquad F : C^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \times C^{2,\alpha}(\partial\Omega)$$

$\|F^{-1}\| \le 1$

## Elliptic Estimates in Holder Spaces

$\partial\Omega \in C^{k+2,\alpha},$ $a_{ij}, b_k \in C^{k,\alpha},$ $\Longrightarrow$

$$\|u\|_{k+2,\alpha} \lesssim \|Lu\|_{k,\alpha} + \|u\|_k$$

*Proof.* For $L = \Delta,$ $k = 1$

$$\|\partial u\|_{2,\alpha} \lesssim \|\Delta\partial u\|_\alpha + \|\partial u\|_C \lesssim \|\Delta u\|_{1,\alpha} + \|u\|_1$$

□

## Elliptic Regularity in Holder spaces

$\partial\Omega \in C^{k+2,\alpha},$ $a_{ij}, b_k, f \in C^{k,\alpha},$ $g \in C^{k+2,\alpha}(\partial\Omega),$ $u \in C^2(\Omega) \cap C(\overline{\Omega})$

$$\begin{cases} Lu = f & in\ \Omega \\ u = g & on\ \partial\Omega \end{cases} \Longrightarrow u \in C^{k+2}(\overline{\Omega}).$$

*Proof.* For $L = \Delta + Id,$ $k = 1$ : if $v = \partial_i u \in C^2$, it should satisfy

$$\begin{cases} \Delta v = \partial_i f - \partial_i u & \in\ \Omega \\ v = \partial_i g & on\ \partial\Omega \end{cases} \tag{44}$$

solving for $v$, we get $v \in C^{2,\alpha}(\overline{\Omega})$. Let $\delta_h u$ be finite difference approximation of $\partial_i u$

$$\delta_h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

$$\Delta\delta_h u = \delta_h f - \delta_h u \tag{45}$$

Subtract (45) from (44)

$$\begin{cases} \Delta(v - \delta_h u) = \partial_i f - \delta_h f + \delta_h u - \partial_i u & in\ \Omega \\ v - \delta_k u = \partial_i g - \delta_h u & on\ \partial\Omega \end{cases}$$

$\|v - \delta_h u\|_\infty \lesssim \|\partial_i f - \delta_h f\|_\infty + \|\partial_i g - \delta_h u\|_\infty \to 0.$ □

Meaning: solution is as regular as the data allows. In particular, if everything is smooth, then solution is smooth. E.g

$$\begin{cases} Lu - \lambda u = 0 & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}$$

and $L\ is\ C^\infty,\ \partial\Omega \in C^\infty \implies u \in C^\infty$.

### Interior Regularity

$a_{ij}, b_k, f \in C^{k,\alpha},\ u \in C^2$. $Lu = f$ in $\overline{B_{3r}} \subset \Omega \implies u \in C^{k+2,\alpha}(B_r)$.

*Proof.*

$$\begin{cases} \Delta v = \partial_i f - \partial_i u & in\ B_{2r} \\ v = \partial_i u & on\ \partial B_{2,r} \end{cases} \qquad \implies v \in C^{3,\alpha}(\overline{B_r}).$$

$\Delta\delta_h u = \delta_h f - \delta_h u$ in $B_{2r}$

$$\implies \|u - \delta_h u\|_\infty \lesssim \|\partial_i f - \delta_h f\|_\infty + \|\partial_i u - \delta_h u\|_\infty + \|\partial_i u - \delta_h u\|_{C(\partial\Omega)}.$$

□

Case $b_0 > 0$: In general, $L$ is not invertible with homogeneous B.C, but $L - \lambda Id$ is invertible for sufficiently large $\lambda_0 > 0$. $\lambda - \kappa_\lambda = (L - \lambda Id)^{-1}$ is called the Resolvent of $L$. When it exists,

$$\kappa_\lambda : C^\alpha(\overline{\Omega}) \to C^{2,\alpha}(\overline{\Omega}) \subset\subset C^\alpha(\overline{\Omega})$$

so $\kappa_\lambda : C^\alpha \to C^\alpha$ is compact.

$$(L - \lambda)u = f - \lambda u \Leftrightarrow u = \kappa_\lambda f - \lambda\kappa_\lambda u \Leftrightarrow (Id - \lambda\kappa_\lambda)u = \kappa_\lambda f$$

# Lecture 22

$F = Id + \lambda k_\lambda$ is Fredholm ie. $k = dimker(F) < \infty$ $r = codimrange(F) < \infty$.

**Theorem 16.** *$f \in RangeF$ if and only if $< f, v >= 0$ for all $v \in KerF^*$. Index $F = k - r$*

### Direct Ferdholm approximation:

**Theorem 17.** *$X \subset\subset Y$, $A : X \to Y$ bounded linear, $\exists c > 0$ such that*

$$\|X\|_X \leq C(\|Ax\|_Y + \|X\|_Y) \qquad \forall x \in X.$$

*$\implies dimkernelA < \infty$ range of $A$ is closed.*

$Au = (Lu, \partial_\nu u)$

$$A : C^{2,\alpha}(\overline{\Omega}) \to C^\alpha(\overline{\Omega}) \times C^{1,\alpha}(\partial\Omega)$$

$$\|u\|_{2,\alpha} \approx \|Lu\|_\alpha + \|\partial_\nu u\|_{1,\alpha;\partial\Omega} + \|u\|_\alpha$$

Neuman-Laplacian is Fredholm of index $O$.

### Spectral Theorem for Compact Operators

$X$ infinite dimensional Banach space, $K : X \to X$ compact. $\exists \{\mu_k\} \subset \mathbb{C}$ s.t $\mu_k \to 0$ ($\mu_0 = 0$) s.t:

$(\mu Id - K)^{-1}$ bounded if $\mu \neq \mu_k$ for all $k$.
$1 \leq dimKer(\mu Id - k) < \infty$ for all $k \geq 1$
$1 \leq dimker(K)$

## Lecture 23— The Heat Equation

Consider

$$\partial_t u = \Delta u : \; y_y = Ay$$

Heuristic argument: Parabolic equations are infinite dimensional limit of elliptic equations.

$$\Delta u = \partial_r^2 u + \frac{N-1}{r}\partial_r u + \Delta_z u$$

taking $N \to \infty$. More precisely,

$$\tau = \frac{r^2}{N}, \qquad \partial_r = \frac{2r}{N}\partial_\tau$$

$$\partial_r^2 = \frac{4r^2}{N^2}\partial_\tau^2 + \frac{2}{N}\partial_\tau$$

$$\begin{aligned} \Delta u &= \frac{4r^2}{N^2}\partial_\tau^2 u + \frac{2}{N}\partial_\tau u + \frac{2N-2}{N}\partial_\tau u + \Delta_z u \\ &= \frac{4\tau}{N}\partial_\tau^2 u + 2\partial_\tau u + \Delta_z u \end{aligned}$$

$N \to \infty (t = -\tau)$.

Cauchy problem

$$\begin{cases} \partial_t u = \Delta u & in \ \mathbb{R}^n \times \mathbb{R}_+; \ \mathbb{R}_+ = \{x > 0\} \\ u = g & on \ \mathbb{R}^n \end{cases}$$

Heat kernel:

$$E(x,t) = \theta(t)(4\pi t)^{-n/2} e^{-|x|/4t}$$

$E \in C^\omega(\mathbb{R}^n \times \mathbb{R}_+)$. Equation is invariant under transformation:

$$t \mapsto \lambda^2 t$$

$$x \mapsto \lambda t$$

$$u(x,t) = \phi(|x|^2/t)$$

find radial solution for ODE to find kernel.

**Definition 6.**

$$(e^{t\Delta} g)(x) := \int\limits_{\mathbb{R}^n} E(x-y,t) g(y) d^n y$$

*Heat propagator.*

**Theorem 18.** *$g \in C_b(\mathbb{R}^n)$, define $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by heat propagator*

$$u(x,t) = (e^{t\Delta} g)(x)$$

*then $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ and $\partial_t u = \Delta u$ in $\mathbb{R}^n \times \mathbb{R}_+$. Moreover,*

$$e^{t\Delta} g \to g \ locally \ uniformly \ as \ t \to 0 \ and$$

$$\|u\|_\infty \leq \|g\|_\infty$$

*Proof.*

$$\partial_t E = \Delta E \qquad in \ \mathbb{R}^n \times \mathbb{R}_+$$

We use the fact that

$$\int E(x,t) = 1, \ E \geq 0 \ E(x,t) \to 0 \ as \ t \to 0 \ if x \neq 0$$

let

$$z = \frac{x-y}{2\sqrt{t}} \implies y = x - 2\sqrt{t}z$$

$$d^n y = (2\sqrt{t})^n d^n z$$

$$u(x,t) = (4\pi t)^{-n/2} \int e^{-|z|^2} g(x - 2\sqrt{t}z)(2\sqrt{t})^n d^n z$$

$$\leq \pi^{-n/2} \|g\|_\infty \int e^{-|z|^2} dz = \|g\|_\infty$$

□

$$\partial_t u = \Delta u, \qquad E(x,t) = \theta(t)(4\pi t)^{-n/2} exp(-\frac{|x|^2}{4t})$$

$$\theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$u(x,t) = [e^{t\Delta} g](x) = \int_{\mathbb{R}^n} E(x-y,t) g(y) dy$$

**Theorem 19.** *If $g \in C_b(\mathbb{R}^n)$, then $u \in C^\infty(\mathbb{R}^n \times \mathbb{R}_+)$ and $\partial_t u = \Delta u$ in $\mathbb{R}^n \times \mathbb{R}_+$*

$$e^{t\Delta} g \to g \ as \ t \downarrow 0, \ \|u\|_\infty \leq \|g\|_\infty$$

Some notes on the hear propagator operator:

$g \mapsto u : C_b(R^n) \to C_b(R^n \times R_+)$ is bounded. One needs $|g(x)| \leq Ce^{\alpha|x|^2}$ to ensure integrability under the heat proagator defined above. The formula is useless for $t < 0$. Moreover, the heat propagator has a smoothing effect; irreversible effect. Disturbances propagate at $\infty$ speed.

## Duhamel's Principal

If $u_t = Lu,\ u|_{t=0} = g$ is solved by $u(t) = S(t)g$, then $u_t - Lu = f,\ u|_{t=0} = 0$ is solved by

$$u(t) = \int_0^t S(t-s)f(s)\ ds. \tag{46}$$

otherwise known as Duhamel's formula (DF).

We know $\int_{\mathbb{R}^n} E(x-y,t)g(y)dy$ solves $u_t = \Delta u,\ u|_{t=0} = g$. Duhamel says

$$\int\limits_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(x-y,t-s)f(y,s)\ dyds \tag{47}$$

solves $u_t - \Delta u = f,\ u|_{t=0} = 0,\ (f(y,s) = 0\ \forall s < 0)$.

## Rigorous Treatment

Suppose $\partial_t u - \Delta u = f$ in $\mathbb{R}^n \times (0,T)$.

$$I = [e^{(t-s)\Delta}u(s)](x) = \int_{\mathbb{R}^n} E(x-y,t-s)u(y,s)\ dy; \qquad (0 < s < t < T)$$

assume $u \in C_b(\mathbb{R}^n \times (0,t]$.

$$\frac{\partial I}{\partial s} = -\underbrace{\int_{\mathbb{R}^n} \frac{\partial E}{\partial t} u(y,s)dy}_{\Delta e^{(t-s)\Delta}u(s)} + \int_{\mathbb{R}^n} E\ \underbrace{\frac{\partial u}{\partial s}(y,s)}_{f+\Delta u} dy \tag{48}$$

$$\int E(x-y,t-s)\Delta u(y,s)dy = \int \Delta E \cdot u \tag{49}$$

$$\textit{By Greens II vanishing boundary terms by taking very large ball.} \tag{50}$$

$$Assume\ the\ following\ with\ 0 < a < b < T \tag{51}$$

$$u \in C_x^2(\mathbb{R}^n \times (0,T)), \tag{52}$$

$$\partial_i^2 u \in C_b(\mathbb{R}^n \times [a,b]), \tag{53}$$

$$\partial_t u \in C_b(\mathbb{R}^n \times [a,b]). \tag{54}$$

$$\tag{55}$$

We have

$$e^{(t-b)\Delta}u(b) - e^{(t-a)\Delta}u(a) = \int_a^b \int_{\mathbb{R}^n} E(x-y,t-s)f(y,s)\ dyds$$

$a \downarrow 0,\ b \uparrow t$. RHS holds provided $f \in C_b(\mathbb{R}^n \times [0,t])$

$$e^{(t-a)\Delta}u(a) = \int_{\mathbb{R}^n} E(x-y,t-a)u(y,a)dy \tag{56}$$

$$= \int E(x-y,t-a)u(y,0)dy + \int E(x-y,t-a)[u(y,a) - u(y,0)]dy \tag{57}$$

$$\tag{58}$$

[taking balls to attain local uniformly: more detail] Note that$[u(y,a) - u(y,0)]$ is continuous so $u(y,a) \to u(y,0)$ locally uniformly, but since our integration domain is infinite, we take a compact domain and we have the convergence and outside the ball, we use the decayness of E to ensure the convergence outside the ball

$$e^{(t-b)\Delta}u(b) = \int E(x-y,t-b)u(y,t)dy \tag{59}$$

$$= \int E(x-y,t-b)u(y,t)dy + \int E(x-y,t-b)[u(y,b) - u(y,t)]dy \tag{60}$$

**Theorem 20.** *Let* $u \in C^1(\mathbb{R}^n \times (0,T)) \cap C_x^2(\mathbb{R}^n \times (0,T))$. *Furthermore, assume* $u \in BC(\mathbb{R}^n \times (0,b])$, $\forall b < T$. *Also, assume* $u \in BC^1(\mathbb{R}^n \times [a,b]) \cap BC^2(\mathbb{R}^n \times [a,b])$, $\forall a,b \in (0,T)$. $g \in C_b(\mathbb{R}^n)$.

$$\begin{cases} \partial_t u - \Delta u = f & \mathbb{R}^n \times (0,T) \\ u = g & \mathbb{R}^n \times \{t=0\} \end{cases} \tag{61}$$

*Then,*

$$u(t) = e^{t\Delta}g + \int_0^t e^{(t-s)\Delta}f(s)ds; \qquad (0 < t < T) \qquad [DF]$$

*In particular solution to* (61) *is unique in the class considered.*

**Theorem 21.** $g \in C_b(\mathbb{R}^n)$, $f \in C_b(\mathbb{R}^n \times (0,b]) \cap C_b^1(\mathbb{R}^n \times [a,b])$, $(0 < a < b < T)$. *Then (DF) solves* (61) *on* $(0,T)$.

*Proof.*

$$\partial_t u = \Delta e^{t\Delta}g + f(t) + \int_0^t \Delta e^{(t-s)}f(s)ds \tag{62}$$

$$= f(t) + \Delta\left(e^{t\Delta}g + \int_o^t e^{(t-s)\Delta}f(s)ds\right) \tag{63}$$

$$\tag{64}$$

To justify taking the Laplacian out from under the integral we need the following:

$$e^{-\frac{|x|^2}{4t}} \to e^{-|z|^2/4t} \textit{ by } |x| \; |z|\sqrt{t}$$

$$\partial_x E \approx E \cdot \frac{|x|}{t} \approx E\frac{|z|}{t^{1/2}},$$

$$\int \Delta E \cdot f = \int_0^t \underbrace{\nabla E}_{(t-s)^{-1/2}} \cdot \nabla f$$

∎

# Lecture 24

## Last time— Duhamel's Principal

Essentially a formula to solve the inhomogeneous heat equation: Let $g \in C_b(\mathbb{R}^n)$

$$f \in C_b(\mathbb{R}^n \times (0,t]) \cap C_b^1(\mathbb{R}^n \times [s,t]) \qquad 0 < s < t < T.$$

We have

$$c$$

solves the heat equation

$$\begin{cases} \partial_t u - \Delta u = f & in\ \mathbb{R}^n \times (0,T) \\ u = g & on\ \mathbb{R}^n \times \{0\} \end{cases}$$

## Systems

Let

$$u, f : \mathbb{R}^n \times (0,T) \to \mathbb{R}^m, \qquad g : \mathbb{R}^n \to \mathbb{R}^m$$

and consider

$$\begin{cases} \partial_t u_i - \Delta u_i = f_i \\ u_i = g_i \end{cases} \qquad i = 1, ..., m$$

where explicitly the solution is

$$u_i(t) = e^{t\Delta} g_i + \int_0^t e^{(t-s)\Delta} f_i(s)\ ds, \qquad i = 1, ..., m.$$

we write for simplicity

$$u(t) = e^{t\Delta} g + \int_0^t e^{(t-s)\Delta} f(s)\ ds.$$

We define the $C_b(\Omega, \mathbb{R}^n)$ by

$$\|u\|_\infty = \max_i \|u_i\|_\infty$$

# Nonlinear Reaction-diffusion Equations

$$\begin{cases} \partial_t u - \Delta u = f(u) \\ u = g \end{cases} \quad where\ f : \mathbb{R}^m \to \mathbb{R}^m. \tag{65}$$

## Example

$$\begin{cases} \partial_t - \Delta u = (u-v)^3 \\ \partial_t v - \Delta v = uv \end{cases} \quad where\ f([u,v]^T) = [(u-v)^3, uv]^T$$

We have solution

$$u(t) = e^{t\Delta u} g + \int_0^t e^{(t-s)\Delta} f(u(s))\ ds \Leftrightarrow u = \phi(u) \tag{66}$$

We attempt to iterate since $u_i$ are arguments of $f$ so for $u_1, u_2, ...$ where we hope that $u_k \to u$, with $u$ solving (65).

**Lemma 5.** *Let $g \in C_b(\mathbb{R}^n),\ f \in C^1,\ u \in C_b(\mathbb{R}^n \times (0,T))$ solves (66), then $u$ also solves (65).*

*Proof.* If $u \in C_b^1(\mathbb{R}^n \times (s,t))\ \forall 0 < s < t < T$. Then its obvious because $f \circ u$ satisfies the conditions of Duhamel.

$$\partial_j(e^{(t-s)\Delta} f(u(s))) = O((t-s)^{-1/2}) \qquad [previous lecture argument on Lap]$$

$\implies \partial_j u$ satisfies "Duhmael". We use $X = C_b(\mathbb{R}^n \times (0,T))$ for $T > 0$ to be chosen later [to insure contraction, we aim to use Banach Fixed point Theorem]. For some large $M > 0$ such that

$$U = \overline{B_M} \subset X.$$

$l = e^{t\Delta} g$.

$$\|\phi(u(t)) - l(t)\|_\infty \leq \int_0^t \|e^{(t-s)\Delta} f(u(s))\|_\infty \, ds \tag{67}$$

$$\leq \int_0^t \|f(u(s))\|_\infty \, ds \tag{68}$$

$$\leq t \cdot \alpha \qquad \forall u \in \overline{B_M}, \tag{69}$$

where $\alpha = \sup_{|v| \leq M} |f(v)|$. Now, $u \in \overline{B_M} \subset X$

$$\|\phi(u) - l\|_\infty \leq T\alpha$$

$$\|\phi(u)\|_\infty \leq \|l\|_\infty + T\alpha \leq \|g\|_\infty + T\alpha$$

Choose $M \geq 2\|g\|_\infty$. $T \leq \frac{\|g\|_\infty}{\alpha}$

$$\implies \phi(u) \in \overline{B_M} \ for \ u \in \overline{B_M}$$

which asserts the invariance of the ball. Now we need to assert contaction:

$$\|\phi(u(t)) - \phi(v(t))\|_\infty \leq \int_0^t \|e^{(t-s)\Delta}[f(u(s)) - f(v(s))]\|_\infty \, ds \tag{70}$$

$$\leq \int_0^t \|f(u(s)) - f(v(s))\|_\infty \, ds \tag{71}$$

analogy with the Mean Value Theorem

$$f(a) - f(b) = f'(\xi)(a-b) \qquad \xi \in (a,b)$$

we have

$$f_i(a) - f_i(b) = \nabla f_i(\xi) \cdot (a-b)$$

$\xi = \theta a + (1-\theta)b,\ \theta \in [0,1]$, so returning to our problem

$$\|\phi(u(t)) - \phi(v(t))\|_\infty \leq \int_0^t \beta \|u(s) - v(s)\|_\infty \, ds \leq \beta t \|u - v\|_\infty$$

where

$$\beta = \sup_{|v| \leq M} \max_{i,k} |\partial_i f_k(v)|,$$

$$\implies \|\phi(u) - \phi(v)\|_\infty \leq \beta T \|u - v\|_\infty$$

choose $T < \frac{1}{\beta}$ and we have existence on $(0,T)$. $\square$

**Theorem 22.** *Let $g \in C_b(\mathbb{R}^n),\ f \in C^1$. Then $\exists T > 0$ s.t (65) has a solution $u \in C^1(\mathbb{R}^n \times (0,T)) \cap C_x^2(\mathbb{R}^n \times (0,T))$. It is unique in the class considered in Duhamel. Moreover, unless $|u(x,t)| \to \infty$ as $t \to T$ as some $x \in \mathbb{R}^n$, $u$ can be continued beyond $T$.*

### A Prioi estimates:

If $u$ is a solution on $(0,T)$ then

$$\|u\|_\infty \leq \psi(T)$$

$$\|u\|_{L^2} \leq \psi(t)$$

$$\underbrace{\|\nabla u\|}_{more\ critical} < \infty \qquad for\ some\ \psi : \mathbb{R}_+ \to \mathbb{R}_+$$

$$\|u\| < \infty$$

# Lecture 25

## Maximum Principles

Let $\Omega \subset \mathbb{R}^n$ bounded domain. $0 < T < \infty$.

$$L = \sum_{i,j} a_{ij}\partial_i\partial_j + \sum_i b_i\partial_i + b_0$$

$a_{ij}, b_k \in C(\overline{Q})$, $L$ elliptic. $Q = \Omega \times (0,T)$ and $\partial_o Q = \overline{\Omega} \times \{0\} \cup \partial\Omega \times (0,T)$.

**Theorem 23.** $u \in C(\overline{Q}) \cap C^1_t C^2_x(Q)$

$$\begin{cases} u_t - Lu \leq 0 & in\ Q \\ u \leq 0 & on\ \partial_o Q \end{cases}$$

$\implies \quad u \leq 0\ in\ \overline{Q}$.

*Proof.* Suppose $M < \infty$ such that $\sup_Q |b_o| < M$ and $u > 0$ at some point in $Q$

$$\implies v(x,t) = u(x,t)e^{-Mt}$$

has positive maximums in $Q \cup \Omega \times \{T\}$

$$\partial_t v \geq 0, \qquad \partial_i v = 0, \qquad \sum a_{ij}\partial_i\partial_j v \leq 0$$

so

$$\partial_t v - Lv \geq b_0 v > -Mv$$

$$\partial_t v - Lv = \partial_t u \cdot e^{-Mt} - Mue^{-Mt} - e^{-Mt}Lu \tag{72}$$

$$\leq -Mue^{-Mt} = -Mv \tag{73}$$

□

## Uniquness

$$\begin{cases} \partial_t u - Lu = f & in\ Q \\ u = g & on\ \partial_o Q \end{cases}$$

has uniqueness in $C(\overline{Q}) \cap C^1_t C^2_x(Q)$.

## Maximum Principal in $\mathbb{R}^n$

*Figure needed.* $E(x,\tau) = E(ix,\tau) = (4\pi\tau)^{-n/2}e^{|x|^2/4\tau}$ satisfies

$$\partial_\tau F = -\Delta f$$

and

$$\partial_t F = \Delta F$$

Define $v_\epsilon = u - \epsilon F$. Assume:

$$|u(x,t)| \le Me^{\alpha|x|^2}$$

then,

$$v_\epsilon(x,t) \le Me^{\alpha|x|^2} - \frac{\epsilon \cdot e^{|x|^2/4(\tau+\delta)}}{[4\pi(\tau+\delta)]^{n/2}} \le \ if\ |x|\ large.$$

if $u \le 0$ at $\{t=0\}$ then

$$\partial_t v_\epsilon - \Delta v_\epsilon = \partial_t u - \Delta \le 0$$

then by maximum principal we have

$$v_\epsilon \le 0 \quad in\ \mathbb{R}^n \times (0,T)\ \forall \epsilon > 0.$$

so

$$u \le \epsilon F \to 0 \quad as\ \epsilon \to 0.$$

$\square$

## Allen-Cahn

$$\partial_t u = \Delta u + u - u^3$$

—
Consider

$$\partial_t u = \Delta u + u^2$$

compare with

$$\partial_t v = v^2$$

by taking difference $w = u - v$

$$w \ge 0 \implies u \ge v.$$

$$u\partial_t u = \partial_t \frac{|u|^2}{2} = u\Delta u + u^2 - u^4 \tag{74}$$

$$= \frac{1}{2}\Delta|u|^2 - |\nabla u|^2 + u^2 - u^4 \tag{75}$$

since $\nabla(u\nabla u) = |\nabla u|^2 + u\Delta u$

$$\Delta|u|^2 = \nabla^2(u\cdot u) = \nabla(2u\nabla u) = 2|\nabla u|^2 + 2u\Delta u$$